A New Technique for Deriving Electric Fields from Sequences of Vector Magnetograms

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How much information about the magnetic induction equation can one extract from a time sequence of (error-free) vector magnetograms taken in a single layer?

\[
\begin{align*}
\frac{\partial B_x}{\partial t} &= c \left( \frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} \right) \\
\frac{\partial B_y}{\partial t} &= c \left( \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} \right) \\
\frac{\partial B_z}{\partial t} &= c \left( \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} \right)
\end{align*}
\]

Kusano et al. (2002, ApJ 577, 501) stated that only the equation for the normal component of \(B\) (\(B_z\)) can be constrained by sequences of vector magnetograms, because measurements in a single layer contain no information about vertical derivatives. Nearly all current work on deriving flow fields or electric fields make this same assumption. **But is this statement true?**
Can we use the other components of the magnetic induction equation?

To investigate this question, we have found it is useful to use the “poloidal-toroidal” decomposition (henceforth PTD) of the magnetic field and its time derivatives. This formalism has been used extensively in the dynamo community, and in anelastic 3D MHD codes such as ASH and ANMHD to ensure that the magnetic field is solenoidal. Before considering time variability and the induction equation, we first use this formalism to describe the magnetic field:

\[ \mathbf{B} = \nabla \times \nabla \times \beta \mathbf{\hat{z}} + \nabla \times \mathcal{I} \mathbf{\hat{z}} \quad (1) \]

Here, \( \beta \) is the “poloidal” potential, and \( \mathcal{I} \) is the “toroidal” potential. We now show how, starting from a single vector magnetogram, one can derive the potential functions \( \beta, \partial \beta / \partial z \), and \( \mathcal{I} \). Here, we use Cartesian coordinates, but this approach can be done in spherical coordinates as well.

\( \beta \) and \( \mathcal{I} \) have the nice properties that their horizontal laplacians can be directly related to the vertical magnetic field component and the vertical current density:

\[ \nabla_h^2 \beta = -B_z \quad (2); \]

\[ \nabla_h^2 \mathcal{I} = -\frac{4\pi J_z}{c} = -\mathbf{\hat{z}} \cdot (\nabla_h \times \mathbf{B}_h) \quad (3) \]
Now, apply the horizontal divergence operator $\nabla_h$ to the magnetic field $\mathbf{B}$:

$$\nabla_h \cdot \mathbf{B} = \nabla_h \cdot \mathbf{B}_h = \nabla^2_h \left( \frac{\partial \beta}{\partial z} \right) \quad (4)$$

The Poisson equation (4) can be understood physically through the solenoidal constraint on $\mathbf{B}$: The left hand side of equation (4) must be equal and opposite to the vertical derivative of $B_z$. Since the horizontal laplacian of $\beta$ is $-B_z$ it follows that the horizontal laplacian of $\partial \beta/\partial z$ must be $-\partial B_z/\partial z$.

From a 2-D map of the 3 components of $\mathbf{B}$, one can solve the three Poisson equations (2-4) for $\beta$, $\partial \beta/\partial z$, and $\mathcal{J}$, subject to appropriate boundary conditions. Note that some information about vertical derivatives in the solution was obtained from equation (4).
What are the correct boundary conditions for $\mathcal{J}$, $\partial \beta / \partial z$, and $\beta$?

The transverse components of the magnetic field are determined entirely by $\mathcal{J}$ and $\partial \beta / \partial z$. Equation (1) can be re-written

$$\mathbf{B} = \nabla_h \left( \frac{\partial \beta}{\partial z} \right) + \nabla_h \times \mathcal{J} \hat{z} - \nabla_h^2 \beta \hat{z},$$

from which we derive these coupled von-Neumann boundary conditions:

$$\frac{\partial}{\partial n} \left( \frac{\partial \beta}{\partial z} \right) = B_n - \frac{\partial \mathcal{J}}{\partial s}$$  \hspace{1cm} (6)

$$\frac{\partial \mathcal{J}}{\partial n} = -B_s + \frac{\partial}{\partial s} \left( \frac{\partial \beta}{\partial z} \right)$$  \hspace{1cm} (7)

Here, $\partial / \partial n$ denotes derivatives normal to the magnetogram boundary, and $\partial / \partial s$ denotes derivatives along the boundary.
The Boundary conditions for $\beta$ itself do not affect the derived values of the magnetic field $B$, since it only affects the field component $B_z$ through the Poisson equation (2) itself. But boundary conditions for $\beta$ do affect the solution for the vector potential:

$$A = \nabla \times \beta \hat{z} + \mathbf{j} \hat{z} - \nabla \Lambda. \quad (7)$$

Here, $\Lambda$ is a scalar (gauge) potential, left unspecified for now.

**To Summarize:** We have shown exactly how one can take knowledge of the vector magnetic field within a bounded 2-d region, and solve 3 Poisson equations, using the boundary conditions that match the observed magnetic field along the boundaries of the magnetic field map. One can in fact reproduce the input magnetic field after the fact from the solutions of the 3 Poisson equations.

Now, consider what happens when we replace $B$ by its time derivative, $\partial B/\partial t$, in equation (1): All of the formalism we have just done will carry through in exactly the same way – we will derive three Poisson equations, analoguous to equations (2-4). The difference is that the solutions to these Poisson equations contain information about all 3 components of the magnetic induction equation.
On to the induction equation...

Performing the substitution just described, we derive these Poisson equations relating the time derivative of the observed magnetic field $\mathbf{B}$ to corresponding time derivatives of the potential functions:

\[
\nabla_h^2 \mathbf{\beta} = -\dot{\mathbf{B}}_z \quad (8);
\]

\[
\nabla_h^2 \mathbf{J} = -\frac{4\pi \hat{\mathbf{J}}_z}{c} = -\hat{\mathbf{z}} \cdot (\nabla_h \times \dot{\mathbf{B}}_h) \quad (9);
\]

\[
\nabla_h^2 \left( \frac{\partial \mathbf{\beta}}{\partial z} \right) = \nabla_h \cdot \dot{\mathbf{B}}_h \quad (10)
\]

The boundary conditions for equations (9) and (10) are specified by the time derivatives of the horizontal fields at the boundaries. The boundary condition for equation (8) is not constrained by the observed time derivatives of the magnetic fields. These equations and boundary conditions parallel exactly the case for the potentials that describe the magnetic field itself in equations (2-4).

Since the time derivative of the magnetic field is equal to $-c \nabla \times \mathbf{E}$, we can immediately relate the curl of $\mathbf{E}$ and $\mathbf{E}$ itself to the potential functions determined from the 3 Poisson equations:
Relating $\nabla \times E$ and $E$ to the 3 potential functions:

$$\nabla \times E = -\frac{1}{c} \nabla_h \left( \frac{\partial \beta}{\partial z} \right) - \frac{1}{c} \nabla_h \times \mathbf{j} \hat{z} + \frac{1}{c} \nabla^2_h \dot{\beta} \hat{z}$$

(11)

$$E = -\frac{1}{c} \left( \nabla_h \times \dot{\beta} \hat{z} + \mathbf{j} \hat{z} \right) - \nabla \psi = E_t - \nabla \psi$$

(12)

The expression for $E$ in equation (12) is obtained simply by uncurling equation (11). Note the appearance of the 3-d gradient of an unspecified scalar potential $\psi$.

The induction equation can be written in component form to illustrate precisely where the depth derivative terms $\frac{\partial E_y}{\partial z}$ and $\frac{\partial E_x}{\partial z}$ occur:

$$\frac{\partial B_x}{\partial t} = c \left( \frac{\partial E_y}{\partial z} - \frac{\partial E_x}{\partial y} \right) = \frac{\partial}{\partial x} \frac{\partial \beta}{\partial z} + \frac{\partial j}{\partial y}$$

(13)

$$\frac{\partial B_y}{\partial t} = c \left( \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} \right) = \frac{\partial}{\partial y} \frac{\partial \beta}{\partial z} - \frac{\partial j}{\partial x}$$

(14)

$$\frac{\partial B_z}{\partial t} = c \left( \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial x} \right) = -\nabla^2_h \dot{\beta}.$$  

(15)

Note that these terms originate from the horizontal divergence of time derivatives of the horizontal field (see the discussion following equation 4).
Does it work?

First test: From $\partial B_x/\partial t$, $\partial B_y/\partial t$, $\partial B_z/\partial t$ computed from Bill’s RADMHD simulation of the Quiet Sun, solve the 3 Poisson equations with boundary conditions as described, and then go back and calculate $\partial B/\partial t$ from equation (11) and see how well they agree. Solution uses Newton-Krylov technique:
Comparison to velocity shootout case:

\[ \frac{\partial B_x}{\partial t} \text{ ANMHD} \]

\[ \frac{\partial B_y}{\partial t} \text{ ANMHD} \]

\[ \frac{\partial B_z}{\partial t} \text{ ANMHD} \]

\[ \frac{\partial B_x}{\partial t} \text{ derived} \]

\[ \frac{\partial B_y}{\partial t} \text{ derived} \]

\[ \frac{\partial B_z}{\partial t} \text{ derived} \]
Velocity shoot out case (cont’d)

$E_x$

$E_y$

$E_z$

$E_x$ derived

$E_y$ derived

$E_z$ derived
Summary: excellent recovery of $\nabla \times \mathbf{E}$, only approximate recovery of $\mathbf{E}$.

Why is this? The problem is that $\mathbf{E}$, in contrast to $\nabla \times \mathbf{E}$, is mathematically under-constrained. The gradient of the unknown scalar potential in equation (12) does not contribute to $\nabla \times \mathbf{E}$, but it does contribute to $\mathbf{E}$.

In the two specific cases just shown, the actual electric field originates largely from the ideal MHD electric field $-\mathbf{v}/c \times \mathbf{B}$. In this case, $\mathbf{E} \cdot \mathbf{B}$ is zero, but the recovered electric field contains significant components of $\mathbf{E}$ parallel to $\mathbf{B}$. The problem is that the physics necessary to uniquely derive the input electric field is missing from the PTD formalism. To get a more accurate recovery of $\mathbf{E}$, we need some way to add some knowledge of additional physics into a specification of $\nabla \psi$.

We will now show how simple physical considerations can be used to derive constraint equations for $\psi$. 
Deriving constraints on $\psi$:

Dana Longcope, in his development of the MEF velocity inversion technique (Longcope 2004, ApJ 612 1181), described a variational approach for determining a constraint equation for $\psi$. Here, we show how a similar approach might be used, and suggest several possible minimization constraints. If $L(\psi)$ is some functional of $\psi$, minimizing the integral of $L(\psi)$ over a volume near the magnetogram yields this equation:

$$
\frac{d}{dx} \frac{\partial L}{\partial \left( \frac{\partial \psi}{\partial x} \right)} + \frac{d}{dy} \frac{\partial L}{\partial \left( \frac{\partial \psi}{\partial y} \right)} + \frac{d}{dz} \frac{\partial L}{\partial \left( \frac{\partial \psi}{\partial z} \right)} = 0 \quad (16)
$$

For example, suppose $L(\psi) = (\nabla \psi)^2$. This minimizes the energy in the electric field contribution from the potential function. Applying equation (16) yields this constraint equation for $\psi$:

$$
\nabla^2 \psi = 0 \quad (17)
$$
Deriving constraints on $\psi$, cont’d

The MEF formalism of Longcope (2004) minimizes the kinetic energy, $v^2$. If one assumes that $\mathbf{E} = -\mathbf{v} \times \mathbf{B}$, then the functional to be minimized is

$$L(\psi) = \frac{(\mathbf{E}_i - \nabla \psi)^2}{B^2} - \frac{((\mathbf{E}_i - \nabla \psi) \cdot \mathbf{B})^2}{B^4}. \quad (18)$$

Deriving the constraint equation for $\psi$ via equation (16) is tedious but straightforward. One drawback of constraints derived in this way is that the constraint equations frequently involve depth derivatives of unknown quantities.
Boundary conditions for $\mathbf{E}$:

The horizontal Electric field includes contributions from both $\partial \beta / \partial t$ and $\psi$. If boundary values of $\mathbf{E}_h$ are known from physical considerations or can be specified a priori, then this provides the following coupled boundary conditions for $\partial \beta / \partial t$ and $\psi$:

\begin{align}
\frac{1}{c} \frac{\partial \dot{\beta}}{\partial n} &= -E_s - \frac{\partial \psi}{\partial s} \quad (19) \\
\frac{\partial \psi}{\partial n} &= -E_n + \frac{1}{c} \frac{\partial \dot{\beta}}{\partial s} \quad (20)
\end{align}

The numerical techniques needed to apply these boundary conditions should be very similar to those used in applying the boundary conditions for $\partial \beta / \partial z$ and $\mathcal{J}$ that are needed to specify $\mathbf{B}_h$ on the boundaries. Thus far, the Newton-Krylov technique used in Abbett’s RADMHD code has been demonstrated to work in applying these boundary conditions to the Poisson equations. In particular, this provides a clear method for specifying zero horizontal electric fields on the boundaries.
Boundary conditions for potential functions: summary

Observed values of $\partial \mathbf{B}_h / \partial t$ at magnetogram boundary completely specify boundary conditions for $\partial \mathcal{I} / \partial t$ and $(\partial / \partial t) (\partial \beta / \partial z)$.

Imposed boundary conditions for $\mathbf{E}_h$ completely determine boundary conditions for $\partial \beta / \partial t$ and $\psi$. 
Summary of 3-D Electric Field Inversion

Given the knowledge of the magnetic field vector and its time derivative at a single time, in a closed 2-d region, we can derive an electric field whose curl will provide the observed time derivative of $\mathbf{B}$.

However, the electric field thus derived is not uniquely specified. The gradient of a scalar potential can be added to the electric field without affecting its curl or the time evolution of $\mathbf{B}$.

Additional physical constraints on the electric field can be given by specifying an equation that the scalar potential must obey.

If one wishes to impose an ideal electric field $\mathbf{E}=-\mathbf{v}/c \times \mathbf{B}$, a completely separate approach can be used to derive the velocity field $\mathbf{v}$ (see paper by Welsch et al, poster 50).