The Uniqueness of Helicity Flux Density

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ABSTRACT

It has been claimed that the density of helicity flux across a surface is not
gauge-invariant, and therefore that only integrals of the helicity flux are physically
meaningful. Here, I evaluate this claim.

1. Background & Problem Statement

Suppose two horizontal flux-transport velocities, \( u \) and \( u' \), have been derived that are consis-
tent with observed magnetic evolution in a plane over some interval \( \Delta t \) that obeys the
ideal induction equation’s normal component,

\[
\partial_t B_z = -\nabla_h \cdot (u B_z) = -\nabla_h \cdot (u' B_z) ,
\]

where Cartesian geometry is assumed, \( \hat{z} \) is normal to the plane, and \( \nabla_h = [\partial_x, \partial_y, 0]^T \). A
Helmholtz decomposition of both flux transport rates, \( u B_z \) and \( u' B_z \), gives

\[
u B_z = -\nabla_h \chi - \nabla_h \times \psi \hat{z} \]
\[
u' B_z = -\nabla_h \chi' - \nabla_h \times \psi' \hat{z} .
\]

(We need to check boundary conditions on uniqueness of Helmholtz decomposition; see
Arfken.) Here, \( \chi \) & \( \chi' \) are inductive potentials, and \( \psi \) & \( \psi' \) are electrostatic potentials.
Equation (1) implies \( \chi \) and \( \chi' \) can differ by a harmonic function, but does not constrain the
electrostatic potentials at all. We assume that \( \chi = \chi' \), but that

\[
\delta \psi = \psi - \psi' \neq 0 ,
\]

which leads to some non-zero difference in the flux transport rates of

\[
\delta u B_z = \nabla_h \times \delta \psi \hat{z} \neq 0 .
\]

In terms of the flux transport velocity, the flux of magnetic helicity across the \( z = 0 \) plane
is given by (Démoulin and Berger 2003; see also Pariat et al. 2005)

\[
\dot{H}_A = -2 \int da \, [A_\perp \cdot (u B_z)] ,
\]
where \( \mathbf{z} \cdot \nabla_h \times \mathbf{A}_P = B_z \) and \( \nabla_h \cdot \mathbf{A}_P = \mathbf{z} \cdot \mathbf{A}_P = 0 \). This implies that the difference in helicity fluxes, \( \delta_u \hat{H}_A \), due to \( \delta \mathbf{u} B_z \) is

\[
\delta_u \hat{H}_A = -2 \int da \left[ \mathbf{A}_P \cdot (\delta \mathbf{u} B_z) \right] \\
= 2 \int da \left[ \nabla_h \cdot (\delta \mathbf{u} \mathbf{z}) \right] \\
= 2 \int da \left[ \nabla_h \cdot (\delta \mathbf{u} \mathbf{z} \times \mathbf{A}_P) - \delta \mathbf{u} \mathbf{z} \cdot \left( \nabla_h \times \mathbf{A}_P \right) \right].
\]

(7)  (8)  (9)

We now ask the question, “What effect does changing gauge have on \( \delta_u \hat{H}_A \)?”

2. Analysis

To proceed, we add in a gauge-difference field, \( \delta \mathbf{A} = -\nabla_h \Lambda \), and investigate the resulting helicity variation,

\[
\delta_A \delta_u \hat{H}_A = 2 \int da \left[ \nabla_h \cdot (\delta \mathbf{u} \mathbf{z} \times \delta \mathbf{A}) - \delta \mathbf{u} \mathbf{z} \cdot \left( \nabla_h \times \delta \mathbf{A} \right) \right] \\
= 2 \int da \left[ \nabla_h \cdot (\delta \mathbf{u} \mathbf{z} \times \delta \mathbf{A}) \right],
\]

(10)  (11)

where the curl in the second term on the right-hand side of equation (10) vanishes since the gauge difference is the gradient of a scalar. The remaining integration can be rewritten

\[
\delta_A \delta_u \hat{H}_A = 2 \oint d\ell \ \mathbf{n}_h \cdot (\delta \mathbf{u} \mathbf{z} \times \delta \mathbf{A}),
\]

(12)

where \( \mathbf{n}_h \) is the horizontal vector normal to the curve that bounds the area of integration. If \( \psi \) and \( \mathbf{A} \) are suitably localized or periodic, this line integral can vanish, meaning the helicity flux would be gauge-invariant.

But what about the helicity flux density? It would seem that even if the helicity flux from equation (12) vanishes, different choices of gauge could cause different helicity densities. The integrand in equation (10) is proportional to

\[
\mathbf{z} \cdot (\nabla_h \delta \psi \times \delta \mathbf{A}),
\]

(13)

which, in general, will vary for different choices of \( \delta \mathbf{A} \). This variation could lead to variations in computed correlations between different helicity density estimates and the “true” helicity density.
REFERENCES
